1.4. Transform-Domain Representation of Discrete Signals and LDTS

1.4.1. Z -Transform

The z – transform of a discrete-time signal $x(n)$ is defined as the power series

$$X(z) = \sum_{k=-\infty}^{\infty} x(n) z^{-k}$$

where z is a complex variable. The above given relation is sometimes called *the direct z- transform* because it transforms the time-domain signal $x(n)$ into its complex-plane representation $X(z)$. Z - transform of a signal $x(n)$ will be denoted by

$$X(z) = Z[x(n)]$$

Since z – transform is an infinite power series, it exists only for those values of z for which this series converges. The *region of convergence* of $X(z)$ (ROC) is the set of all values of z for which $X(z)$ attains a finite value.

The procedure for transforming from z – domain ($X(z)$) to the time- domain is called the inverse z – transform. It can be shown that inverse z – transform is given by

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

where C denotes the closed contour in the ROC of $X(z)$ that encircles the origin.

1.4.2. Transfer Function

A very convenient tool for analysis LDTS is z-transform. This is particularly so when it is possible to provide an input-output description of the system by means of a constant coefficient linear difference equation as

$$y(n) = \sum_{k=0}^{N} b(k) x(n-k) - \sum_{k=1}^{M} a(k) y(n-k)$$

Application of the z-transform to this equation under zero initial conditions, leads to the notion of *a transfer function*. If

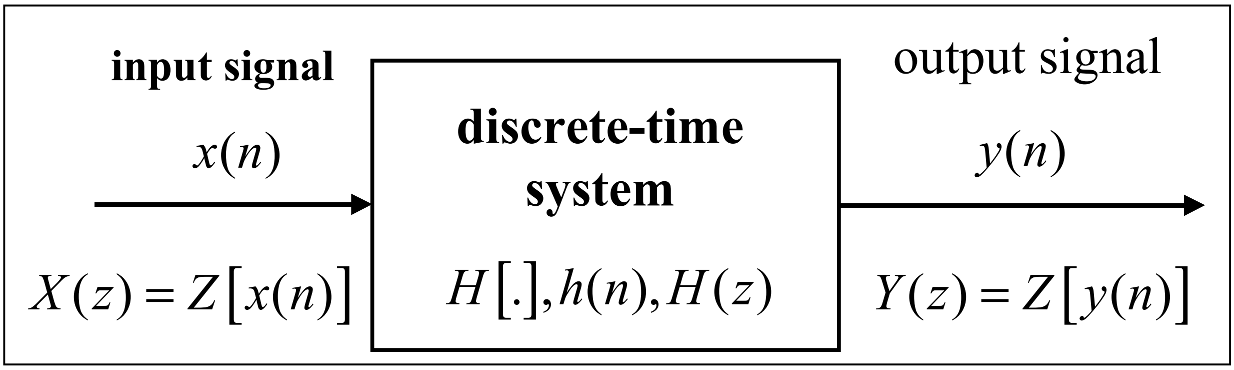
$$Y(z) = Z[y(n)] = \sum_{k=-\infty}^{\infty} y(n) z^{-k} \quad \text{and} \quad X(z) = Z[x(n)] = \sum_{k=-\infty}^{\infty} x(n) z^{-k}$$

denote the z-transforms of the output and input sequence $y(n)$ and $x(n)$, then under zero initial conditions the above given equation are transformed to the following equation:

$$Y(z) = \sum_{k=0}^{N} b(k) z^{-k} X(z) - \sum_{k=1}^{M} a(k) z^{-k} Y(z)$$

The ratio of the output sequence transform and input sequence transform defines the transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{Z[y(n)]}{Z[x(n)]}$$



Then, the transfer function of the LDTS described by the above equation is given by

$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{1 + \sum_{k=1}^{M} a(k) z^{-k}}$$

It follows from this equation that the transfer function $H(z)$ may be viewed as a rational function in the complex variable z^{-1}.

1.4.3. Poles, Zeros, Pole-Zero Plot

Without any loss of generality, it may be assumed that that $H(z)$ has been expressed in its irreducible form; i.e. the numerator and denominator polynomials of $H(z)$ do not have any non-constant common factor. Then $H(z)$ can alternatively be expressed in the form

$$H(z) = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{1 + \sum_{k=1}^{M} a(k) z^{-k}} = \frac{b_0}{a_0} z^{N-M} \frac{\prod_{k=1}^{N} (z - z_k)}{\prod_{k=1}^{M} (z - p_k)}$$

The set $\{z_k\}$ of values of z for which $H(z_k) = 0$ are called *the zeros of* $H(z)$. The set $\{p_k\}$ of values of z for which $H(p_k) \to \infty$ are called *the poles of* $H(z)$. The plot of the zeros and poles of $H(z)$ (called *pole-zero plot*) in the z-plane represents a strong tool for LDTS description.

Example: the 4-th order Butterworth low-pass filter, cut off frequency .

$$b = [\, 0.0186 \quad 0.0743 \quad 0.1114 \quad 0.0743 \quad 0.0186 \,]$$

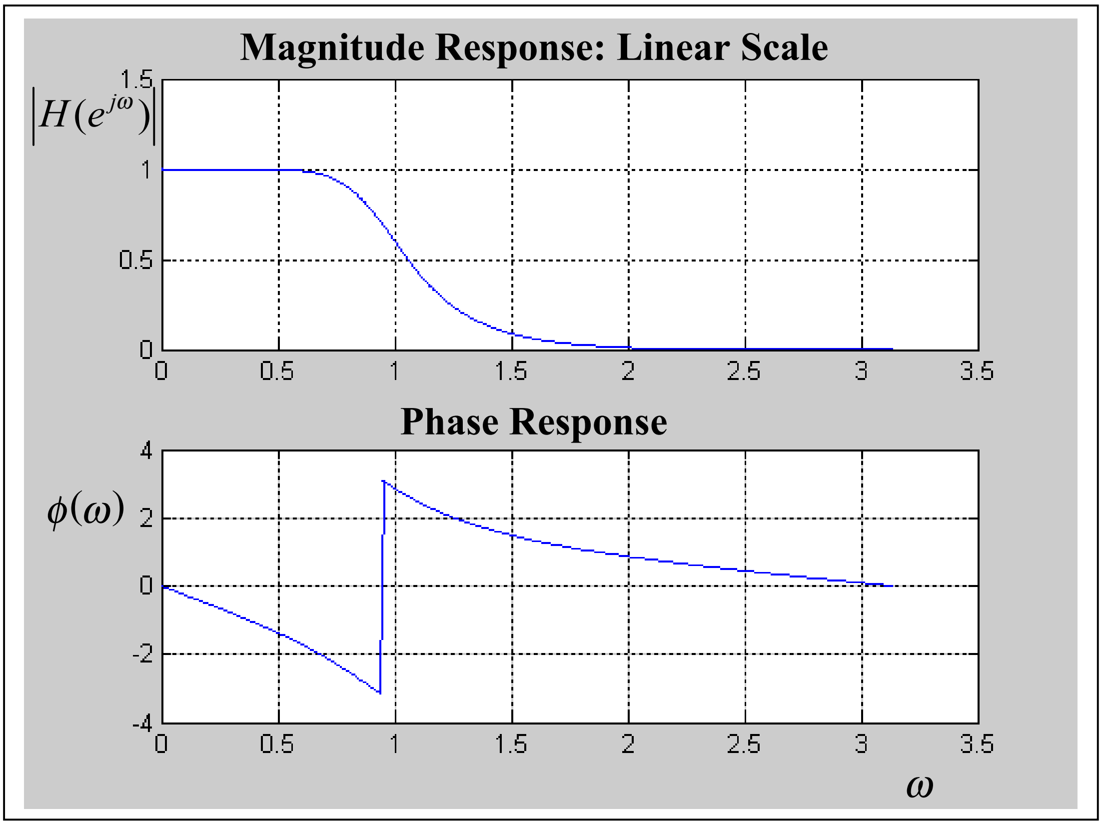
$$a = [\, 1.0000 \quad -1.5704 \quad 1.2756 \quad -0.4844 \quad 0.0762 \,]$$

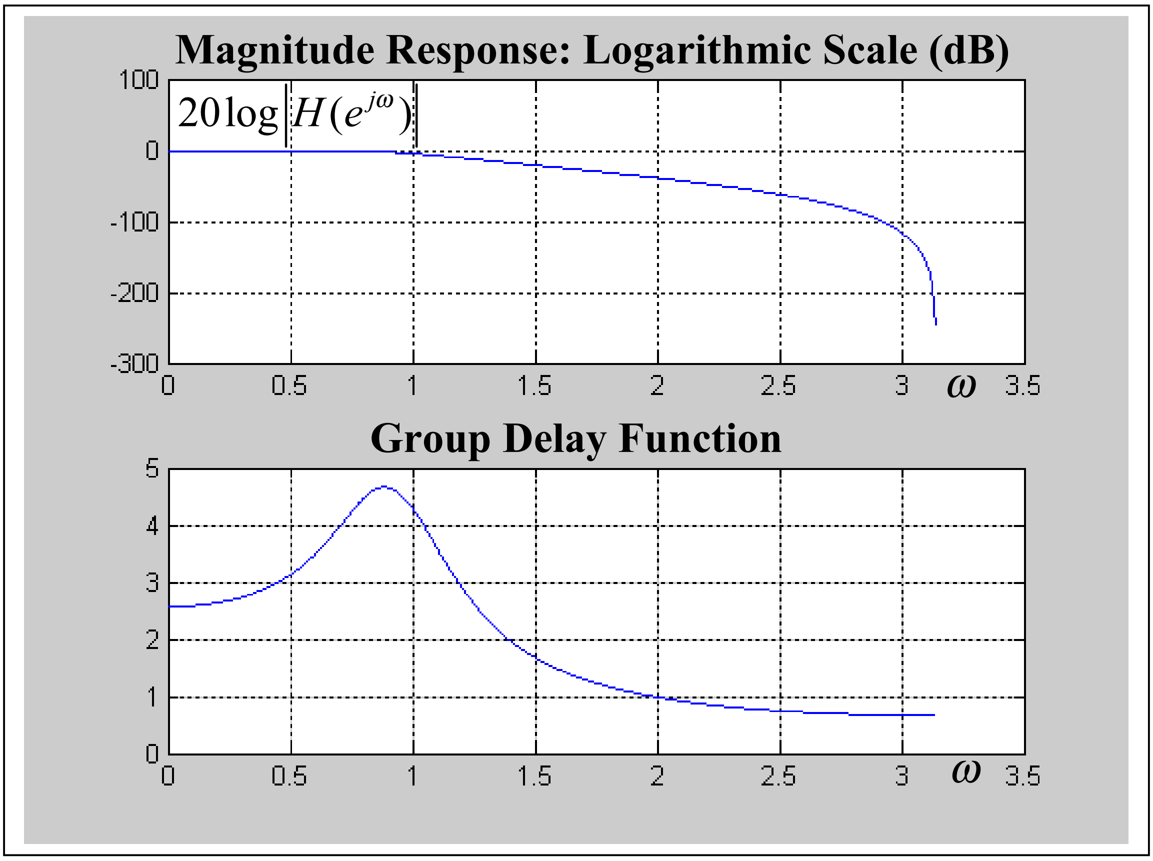
$z_1 = -1.0002$, $z_2 = -1.0000 + 0.0002j$

$z_3 = -1.0000 - 0.0002i$, $z_4 = -0.9998$

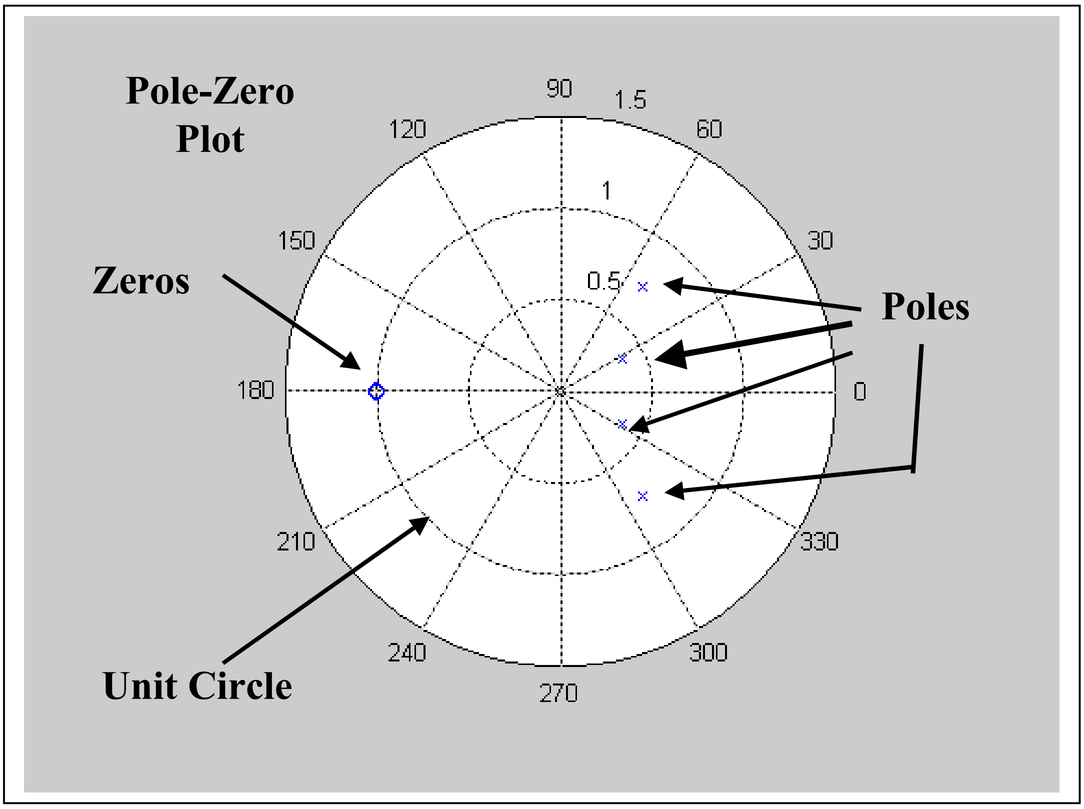
$p_1 = 0.4488 + 0.5707j$, $p_2 = 0.4488 - 0.5707j$

$p_3 = 0.3364 + 0.1772j$, $p_4 = 0.3364 - 0.1772j$

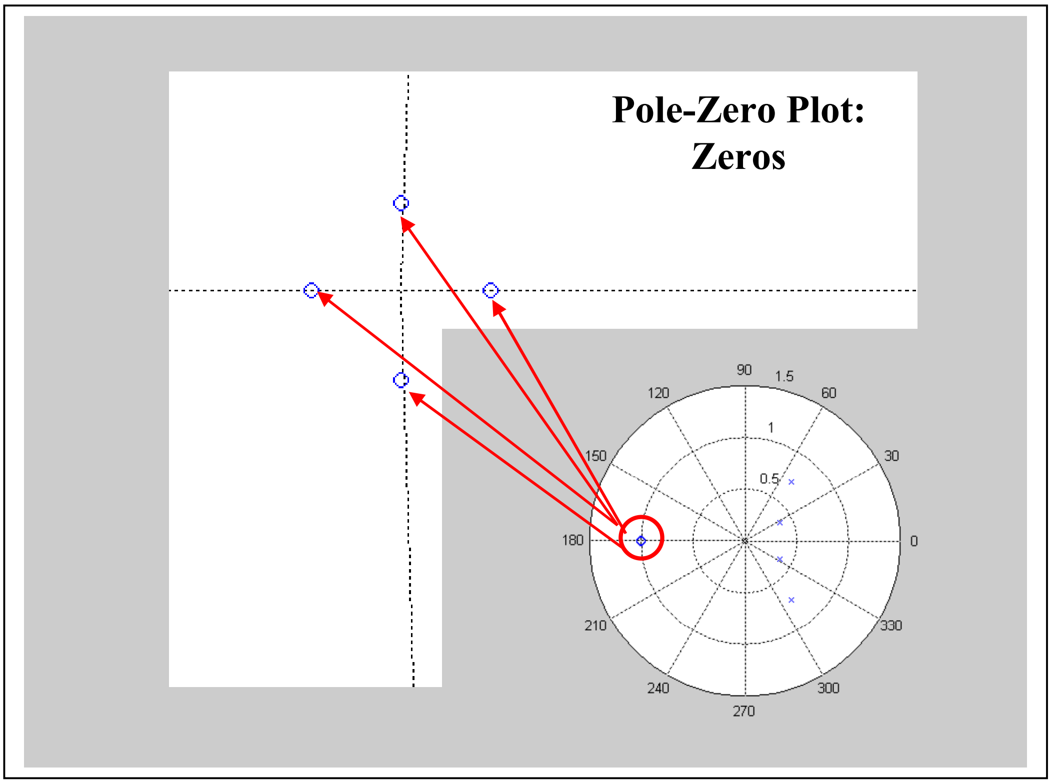




Magnitude Response: Logarithmic Scale (dB)
$20\log|H(e^{j\omega})|$
100
0
-100
-200
-300
0
0.5
1
1.5
2
2.5
3
ω
3.5
Group Delay Function
5
4
3
2
1
0
0
0.5
1
1.5
2
2.5
3
ω
3.5



Pole-Zero Plot
Zeros
Poles
Unit Circle
90
1.5
60
120
1
30
150
0.5
180
0
210
330
240
300
270



1.4.4. Stability of LTI System

LDTS that may be represented, under zero initial conditions, by transfer function

$$H(z)=\frac{\sum_{k=0}^{N} b(k)z^{-k}}{1+\sum_{k=1}^{M} a(k)z^{-k}}=\frac{b_0}{a_0}z^{N-M}\frac{\prod_{k=1}^{N}(z-z_k)}{\prod_{k=1}^{M}(z-p_k)}$$

LDTS is BIBO stable if and only if the unit circle, $|z|=1$, falls within the region of convergence of the power series expansion for its transfer function. In the case when the transfer function characterizes a causal LDTS, the stability condition is equivalent to the requirement that *its transfer function* $H(z)$ *has all its poles inside the unit circle*.

Example 1:

$$H(z)=\frac{1-0.9z^{-1}+0.18z^{-2}}{1-0.8z^{-1}+0.64z^{-2}} \qquad \begin{matrix} z_1=0.3 & p_1=0.4000+0.6928j & |p_1|=0.8<1 \\ z_2=0.6 & p_2=0.4000-0.6928j & |p_2|=0.8<1 \end{matrix}$$

: stable system

Example 2:

$$H(z)=\frac{1-0.16z^{-2}}{1-1.1z^{-1}+1.21z^{-2}} \qquad \begin{matrix} z_1=0.4 & p_1=0.5500+0.9526j & |p_1|=1.1>1 \\ z_2=-0.4 & p_2=0.5500-0.9526j & |p_2|=1.1>1 \end{matrix}$$

: unstable system

1.4.4. Stability of LTI System

LDTS that may be represented, under zero initial conditions, by transfer function

$$H(z)=\frac{\sum_{k=0}^{N} b(k)z^{-k}}{1+\sum_{k=1}^{M} a(k)z^{-k}}=\frac{b_0}{a_0}z^{N-M}\frac{\prod_{k=1}^{N}(z-z_k)}{\prod_{k=1}^{M}(z-p_k)}$$

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Example 2:

$$H(z)=\frac{1-0.16z^{-2}}{1-1.1z^{-1}+1.21z^{-2}} \qquad \begin{matrix} z_1=0.4 & p_1=0.5500+0.9526j & |p_1|=1.1>1 \\ z_2=-0.4 & p_2=0.5500-0.9526j & |p_2|=1.1>1 \end{matrix} \text{ : unstable system}$$

2. Introduction to Digital Filters

2.1. Definitions of Basic Terms

Filtering:	Process of extraction of desired signal from noise
Filter:	system performing filtering
Analogues filtering:	filtering performed on continuous-time signals and yields continuous signals
Digital filtering:	filtering performed on discrete-time signals and yields discrete signals

Examples of filtering operations include:

1. Noise suppression. Examples of signals that are typically noisy include:
- Received radio signals.
- Signals received by imaging sensors, such as television cameras or infrared imaging devices.
- Electrical signals measured from the human body (such as brain, heart, or neurological signals).

2. Enhacement of selected frequency range. Examples:
- Treble and bass control or graphic equalizers in audio systems.
- Enhancement of edges in images.

3. Bandwith limiting.
- Bandwidth limiting as a means of aliasing prevention in sampling.
- Application in FDMA communication systems (**F**requency **D**ivision **M**ultiple **A**ccess - FDMA)

4. Removal or attenuation of specific frequencies.
- Blocking of the DC component of a signal.
- Attenuation of interference from powerline.

5. Special operations.

Operation	***Time Domain / Impulse Response***[*]	***Frequency Domain / Frequency Response***[*]
Differentiation	$y(t) = \frac{dx(t)}{dt}$	$Y(j\omega) = j\omega\, X(j\omega)$
Integration	$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$	$Y(j\omega) = \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$
Hilbert transform[*]	$h(t) = \frac{1}{\pi t}$	$H(j\omega) = -j\, sign(\omega)$

2.2. Filter Specifications

Before a digital filter can be designed and implemented, it is necessary to specify its performance requirements. A typical filter should pass certain frequencies and attenuation other frequencies. Therefore, the frequencies in question, the required gains and attenuations must be defined exactly.

2.2.1. Low-Pass Filters

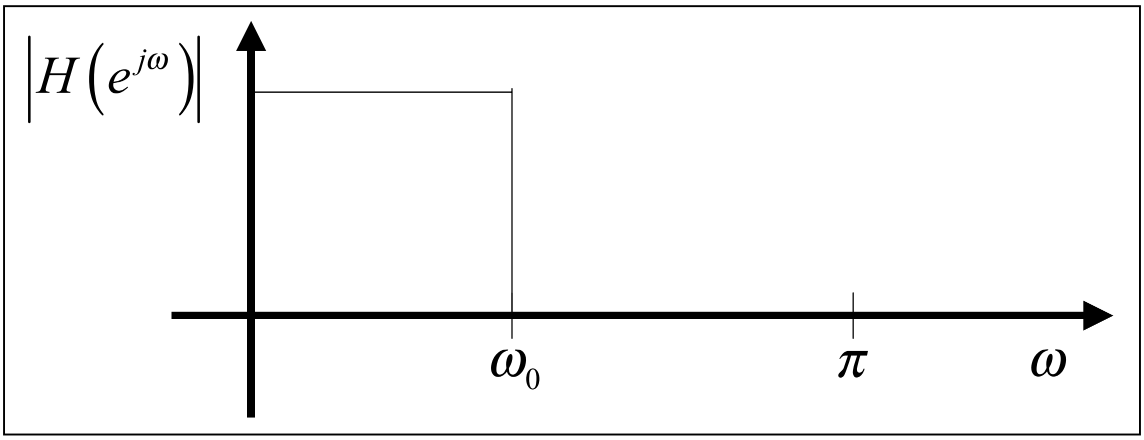
Low-pass filters are designed to pass low frequencies, from zero to a certain out off frequency ω_0, and to block high frequencies.

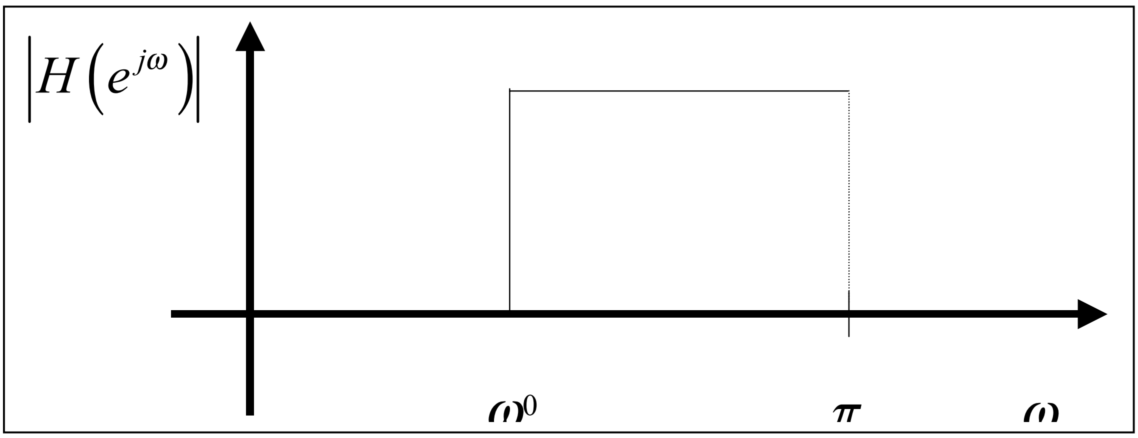
2.2.2. High-Pass Filters

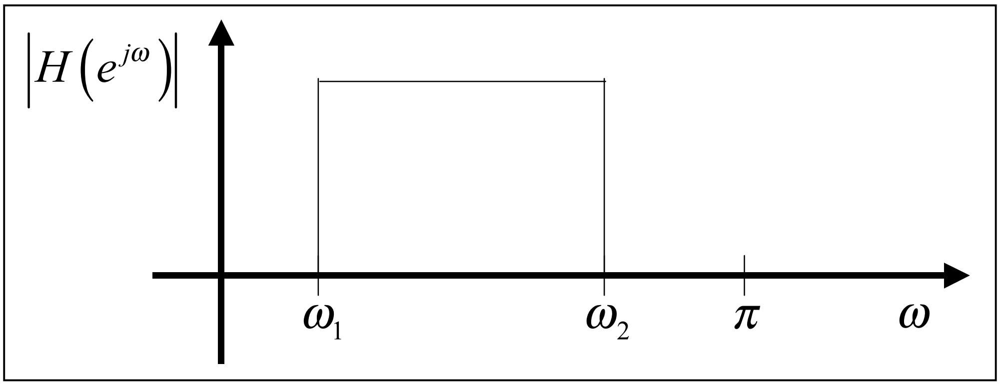
High-pass filters are designed to pass high frequencies, from a certain out off frequency ω_0 to π, and to block low frequencies.

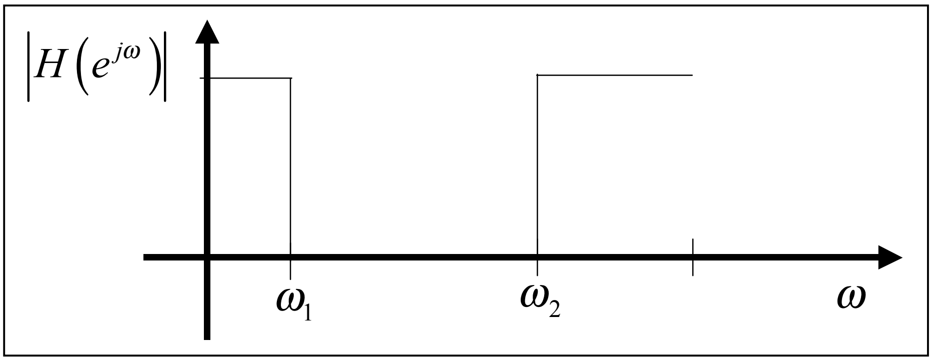
2.2.3. Band-Pass Filters

Band-pass filters are designed to pass a certain frequency range $< \omega_1, \omega_2 >$, which does not include zero, and to block other frequencies.



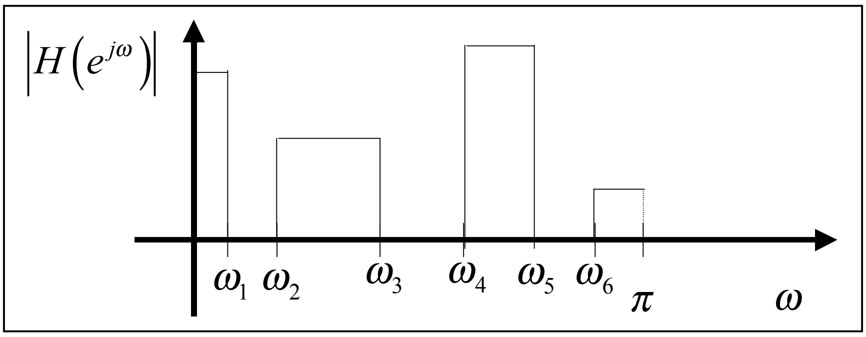






2.2.4. Band-Stop Filters

Band-stop filters are designed to block a certain frequency range $<\omega_1, \omega_2>$, which does not include zero, and to pass other frequencies.



2.2.5. Multiband Filters

The described type of filters are the most commonly used, but they are sometimes far from encompassing the full generality of linear time-invariant filtering. Multiband filters generalize these four types of filters in that they allow for different gains or attenuations in different frequency bands. A piecewise –constant multiband filter is characterized by the following parameters:

a) A division of the frequency range $<0, \pi>$ to a finite union of intervals. Some of these intervals are pass bands, some are stop bands, and the remaining are transition bands.
b) A desired gain and a permitted tolerance for each pass band.
c) An attenuation threshold for each stop band.

2.2.6. All-Pass Filters

A filter is called all-pass if its magnitude response is identically 1 (or, more generally, a positive constant) at all frequencies. The phase response of an all-pass filter is not restricted and is allowed to vary arbitrarily as a function of the frequency.

In general, a rational filter is all-pass if only if it has the same number of poles and zeros (including multiplicities), and each zero is the conjugate inverse of a corresponding pole ($z_k = \frac{1}{p_k}$).

Example:

Transfer function: $H(z) = \frac{0.8 - z^{-1}}{1 - .8z^{-1}}$

2.2.7. Differentiators

A digital differentiator can approximate an analog differentiator only over a limited range of frequencies, determined by the sampling rate. The ideal frequency response of a digital differentiator is therefore

$$H\left(e^{j\omega}\right) = \frac{j\omega}{T}, \quad -\pi \le \omega \le \pi .$$

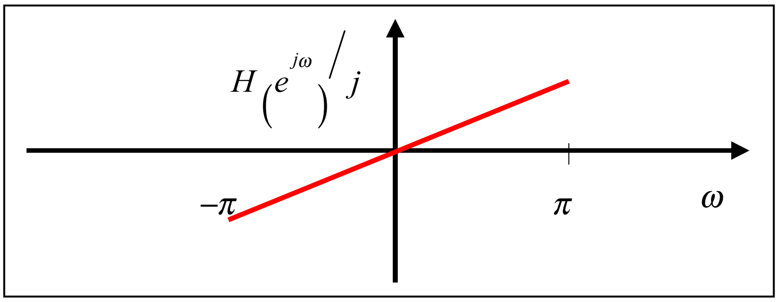
He phase response of an ideal differentiator is 0.5π at all frequencies.

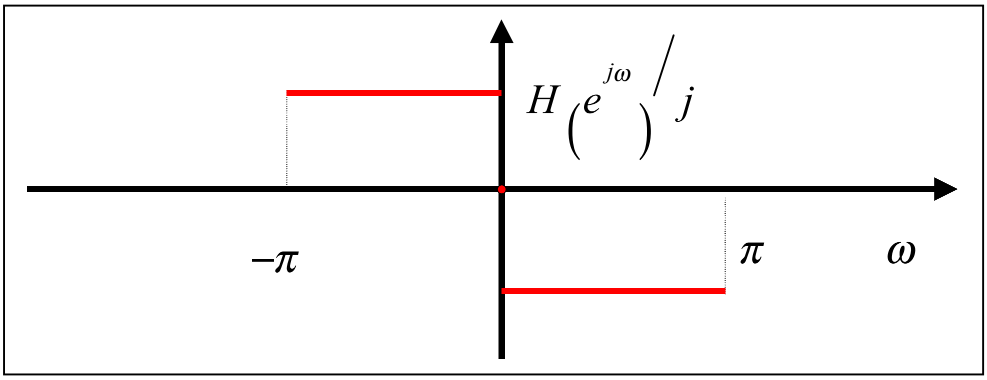
2.2.8. Hilbert Transformers

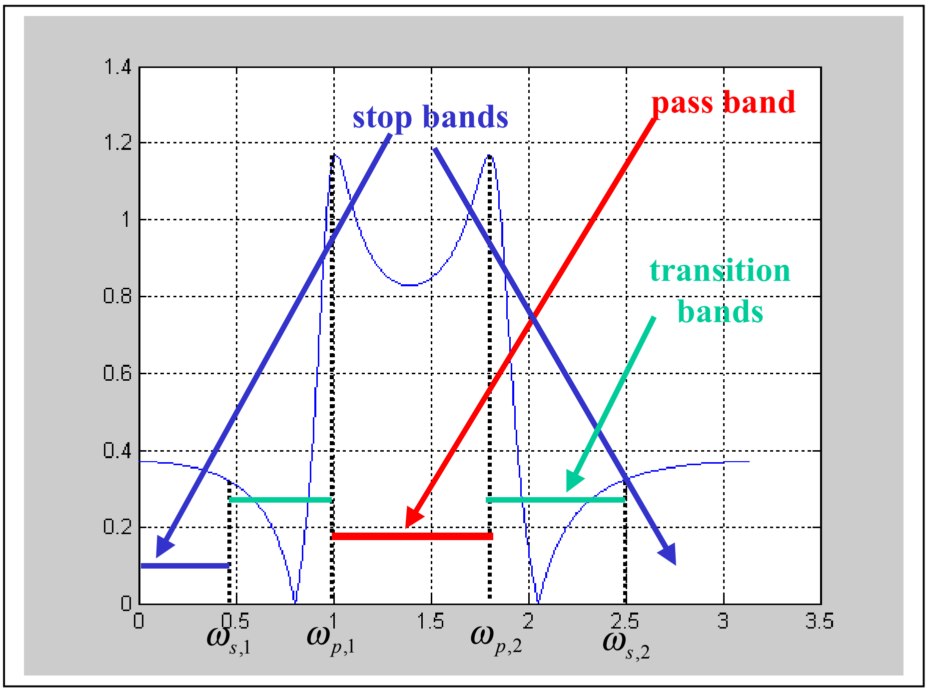
The frequency response of an ideal analog Hilbert transformer is

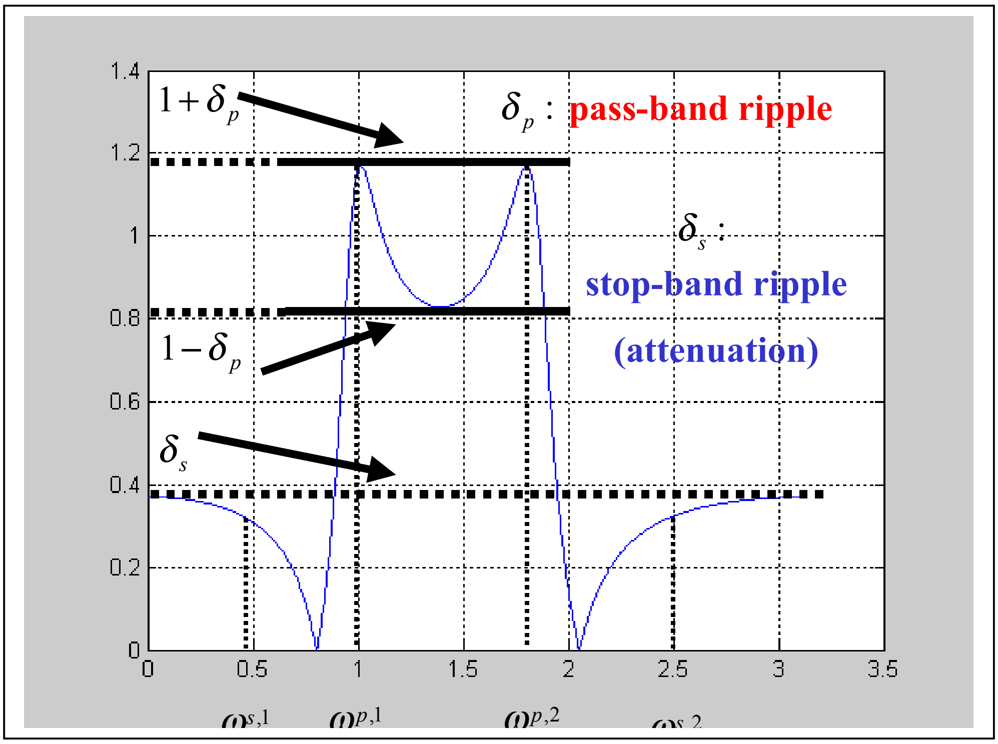
$$H(j\omega) = \begin{matrix} -j & \omega > 0 \\ 0 & \omega = 0 \\ j & \omega < 0 \end{matrix}$$

A digital Hilbert transformer can approximate an analog Hilbert transformer only over a limited range of frequencies, determined by the sampling rate






stop bands
pass band
transition bands
$\omega_{s,1}$
$\omega_{p,1}$
$\omega_{p,2}$
$\omega_{s,2}$


$1+\delta_p$
δ_p : pass-band ripple
δ_s :
stop-band ripple
(attenuation)
$1-\delta_p$
δ_s

3. FIR Digital Filter. Introduction

3.1. FIR Digital Filters. Basic Properties: A Review

Digital FIR filters cannot be derived from analog filters, since rational analog filter cannot have a finite impulse response. In many digital signal processing applications, FIR filters are preferred over their IIR counterparts. The main advantages of the FIR filter designs over their IIR equivalents are the following:

1. FIR filters with exactly linear phase can easily be designed. This simplifies the approximation problem, in many cases, when one is only interested in designing of a filter that approximates an arbitrary magnitude response. Linear phase filters are important for applications where frequency dispersion due to nonlinear phase is harmful - e.g. speech processing and data transmission.
2. There are computationally efficient realizations for implementing FIR filters. These include both non-recursive and recursive realizations.
3. FIR filters realized non-recursively are inherently stable and free of limit cycle oscillations when implemented on a finite-word length digital system.
4. Excellent design methods are available for various kinds of FIR filters with arbitrary specifications.
5. The output noise due to multiplication round off errors in an FIR filter is usually very low and the sensitivity to variations in the filter coefficients is also low.

Among the possible disadvantages of FIR filters are:

1. The relative computational complexity of FIR filter is higher than that of IIR filters. It means, that an FIR filter meeting the same specification as a given IIR filter will require many more operations per unit of time. This situation can be met especially in applications demanding narrow transition bands or if it is required to approximate sharp cut off frequency. As the transitions bandwidth of an FIR filter is made e.g. narrower, the filter order, and correspondingly the arithmetic complexity, increases inversely proportionally to this width. This makes the implementation of narrow transition band FIR filter very costly. The cost of implementation of an FIR filter can, however, be reduced by using multiplier-efficient realizations, fast convolution algorithms, and multirate filtering.
2. The delay of linear phase FIR filters need not always be an integer number of samples. This non-integer delay can lead to problems in some signal processing applications.

3.2. Frequency Response of Linear Phase FIR Digital Filters

One of the simplest types of filters that we can design is an FIR filter with linear phase. It can be shown that only FIR filters can be designed to have linear phase response. Similarly, it can be shown also that IIR filters cannot have linear phase.

An FIR filter of length M described by

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

has a frequency response

$$H(e^{j\omega}) = \sum_{k=0}^{M-1} h(k)e^{-j\omega k}$$

where the filter coefficients $\{h(k)\}$ are also the values of the impulse response of the filter (i.e. $h(k) = 0 \;\; for \;\; k < 0 \;\; and \;\; k > M-1$).

In the next, it will be shown that the linear phase condition is obtained by imposing symmetry conditions on the impulse sample (unit sample response) of the filter. In particular, we consider two different symmetry conditions for $h(k)$, so-called symmetrical impulse response

$$h(k) = h(M-1-k) \quad for \;\; k = 0,1,2,\ldots,M-1$$

and antisymmetrical impulse response

$$h(k) = -h(M-1-k) \quad for \;\; k = 0,1,2,\ldots,M-1.$$

Besides, the length of the impulse response of the FIR filter (M) can be even or odd. Then, the four cases of linear phase FIR filters can be obtained.

3.2.1. Symmetrical impulse response, M even

$k = 0,1,2,\ldots,M-1$

$$h(0) = h(M-1),\; h(1) = h(M-2),\; h(2) = h(M-3),\ldots,\; h(\frac{M}{2}-1) = h(\frac{M}{2}-1)$$

Example:

$M = 4$ (even), symmetrical impulse response:

$$M-1 = 3 \qquad k = 0,1,2,3$$

$$h(0) = h(3) \qquad h(1) = h(2)$$

$$1 = \frac{4}{2} - 1 = \frac{M}{2} - 1$$

End.

$$H(e^{j\omega}) = \sum_{k=0}^{M-1} h(k) e^{-j\omega k} = \sum_{k=0}^{\frac{M}{2}-1} h(k)\left[e^{-j\omega k} + e^{-j\omega(M-1-k)}\right] =$$

$$= 2e^{-j\omega\frac{M-1}{2}} \sum_{k=0}^{\frac{M}{2}-1} h(k) \frac{e^{j\omega\left(\frac{M-1}{2}-k\right)} + e^{-j\omega\left(\frac{M-1}{2}-k\right)}}{2}$$

$$H(e^{j\omega}) = e^{-j\omega\frac{M-1}{2}}\, 2\sum_{k=0}^{\frac{M}{2}-1} h(k)\cos\omega\left(\frac{M-1}{2}-k\right)$$

Here, the real-valued frequency response $H_r(\omega) \in R$ is given by

$$H_r(\omega) = 2\sum_{k=0}^{\frac{M}{2}-1} h(k)\cos\omega\left(\frac{M-1}{2}-k\right)$$

Then,

$$H(e^{j\omega}) = e^{-j\omega\frac{M-1}{2}} H_r(\omega) = \begin{cases} \left|H_r(\omega)\right| e^{-j\omega\frac{M-1}{2}} & for\; H_r(\omega) >= 0 \\ \left|H_r(\omega)\right| e^{-j\left(\omega\frac{M-1}{2}+\pi\right)} & for\; H_r(\omega) < 0 \end{cases}$$

$$\left|H(e^{j\omega})\right| = \left|H_r(\omega)\right|$$

$$\phi(\omega) = \begin{cases} -\omega\dfrac{M-1}{2} & for\; H_r(\omega) >= 0 \\ -\omega\dfrac{M-1}{2} - \pi & for\; H_r(\omega) < 0 \end{cases}$$

We observe that the phase response is a linear function of ω provided that $H_r(\omega)$ is positive or negative. When $H_r(\omega)$ changes the sign from positive to negative (or vice versa), the phase undergoes an abrupt change of π radians. If these phase changes occur outside the pass-band of the filter (i.e., in the stop-band) we do not care, since the desired signal passing through the filter has no frequency content in the stop-band.

3.2.2. Symmetrical impulse response, M odd

Example:

$M = 5$ (odd), symmetrical impulse response:

$M - 1 = 4 \qquad k = 0,1,2,3,4$

$h(0) = h(4) \quad h(1) = h(3) \quad h(2) = h(2)$

$1 = \dfrac{5-3}{2} = \dfrac{M-3}{2} \qquad 2 = \dfrac{M-1}{2}$

End.

$$H(e^{j\omega}) = \sum_{k=0}^{M-1} h(k)e^{-j\omega k} = h\left(\frac{M-1}{2}\right)e^{-j\omega\frac{M-1}{2}} + \sum_{k=0}^{\frac{M-3}{2}} h(k)\left[e^{-j\omega k} + e^{-j\omega(M-1-k)}\right]$$

$$H(e^{j\omega}) = e^{-j\omega\frac{M-1}{2}}\left[h\left(\frac{M-1}{2}\right) + 2\sum_{k=0}^{\frac{M-3}{2}} h(k)\frac{e^{j\omega\left(\frac{M-1}{2}-k\right)} + e^{-j\omega\left(\frac{M-1}{2}-k\right)}}{2}\right]$$

$$H(e^{j\omega}) = e^{-j\omega\frac{M-1}{2}}\left[h\left(\frac{M-1}{2}\right) + 2\sum_{k=0}^{\frac{M-3}{2}} h(k)\cos\omega\left(\frac{M-1}{2} - k\right)\right]$$

Here, the real-valued frequency response $H_r(\omega) \in R$ is given by

$$H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{k=0}^{\frac{M-3}{2}} h(k)\cos\omega\left(\frac{M-1}{2} - k\right)$$

Then,

$$H(e^{j\omega}) = e^{-j\omega\frac{M-1}{2}} H_r(\omega) = \begin{cases} |H_r(\omega)|e^{-j\omega\frac{M-1}{2}} & for\ H_r(\omega) >= 0 \\ |H_r(\omega)|e^{-j\left(\omega\frac{M-1}{2}+\pi\right)} & for\ H_r(\omega) < 0 \end{cases}$$

$$|H(e^{j\omega})| = |H_r(\omega)|$$

$$\phi(\omega) = \begin{cases} -\omega\dfrac{M-1}{2} & for\ H_r(\omega) >= 0 \\ -\omega\dfrac{M-1}{2} - \pi & for\ H_r(\omega) < 0 \end{cases}$$

3.2.3. Antisymmetrical impulse response, M even

Example:

$M = 4$ (even), antisymmetrical impulse response:

$$M-1=3 \qquad k=0,1,2,3$$
$$h(0)=-h(3) \qquad h(1)=-h(2)$$
$$1=\frac{4}{2}-1=\frac{M}{2}-1$$

End.

$$H(e^{j\omega})=\sum_{k=0}^{M-1}h(k)e^{-j\omega k} = \sum_{k=0}^{\frac{M}{2}-1}h(k)\left[e^{-j\omega k}-e^{-j\omega(M-1-k)}\right]=$$
$$=2je^{-j\omega\frac{M-1}{2}}\sum_{k=0}^{\frac{M}{2}-1}h(k)\frac{e^{j\omega\left(\frac{M-1}{2}-k\right)}-e^{-j\omega\left(\frac{M-1}{2}-k\right)}}{2j}$$

$$H(e^{j\omega})=e^{-j\omega\frac{M-1}{2}+\frac{\pi}{2}}2\sum_{k=0}^{\frac{M}{2}-1}h(k)\sin\omega\left(\frac{M-1}{2}-k\right)$$

Here, the real-valued frequency response $H_r(\omega)\in R$ is given by

$$H_r(\omega)=2\sum_{k=0}^{\frac{M}{2}-1}h(k)\sin\omega\left(\frac{M-1}{2}-k\right)$$

$$H_r(0)=2\sum_{k=0}^{\frac{M}{2}-1}h(k)\sin 0\left(\frac{M-1}{2}-k\right)=0$$

Then,

$$H(e^{j\omega})=e^{-j\omega\frac{M-1}{2}+j\frac{\pi}{2}}H_r(\omega)=\begin{cases}\left|H_r(\omega)\right|e^{-j\omega\frac{M-1}{2}+j\frac{\pi}{2}} & for\, H_r(\omega)>=0\\ \left|H_r(\omega)\right|e^{-j\omega\frac{M-1}{2}+j\frac{3\pi}{2}} & for\, H_r(\omega)<0\end{cases}$$

$$\left|H(e^{j\omega})\right|=\left|H_r(\omega)\right|$$

$$\phi(\omega)=\begin{cases}-\omega\frac{M-1}{2}+\frac{\pi}{2} & for\, H_r(\omega)>=0\\ -\omega\frac{M-1}{2}+\frac{3\pi}{2} & for\, H_r(\omega)<0\end{cases}$$

3.2.4. Antisymmetrical impulse response, M odd

Example

$M = 5$ (odd), antisymmetrical impulse response:

$$M - 1 = 4 \qquad k = 0,1,2,3,4$$
$$h(0) = -h(4) \quad h(1) = -h(3) \quad h(2) = -h(2) \rightarrow h(2) = 0$$
$$1 = \frac{5-3}{2} = \frac{M-3}{2} \qquad 2 = \frac{5-1}{2} = \frac{M-1}{2}$$
$$h\left(\frac{M-1}{2}\right) = 0 \text{ !!!!!}$$

$$H(e^{j\omega}) = \sum_{k=0}^{M-1} h(k)e^{-j\omega k} = \sum_{k=0}^{\frac{M-3}{2}} h(k)\left[e^{-j\omega k} - e^{-j\omega(M-1-k)}\right] =$$
$$= 2je^{-j\omega\frac{M-1}{2}} \sum_{k=0}^{\frac{M-3}{2}} h(k) \frac{e^{j\omega\left(\frac{M-1}{2}-k\right)} - e^{-j\omega\left(\frac{M-1}{2}-k\right)}}{2j} =$$
$$= e^{-j\omega\frac{M-1}{2}+\frac{\pi}{2}} 2\sum_{k=0}^{\frac{M-3}{2}} h(k)\sin\omega\left(\frac{M-1}{2}-k\right)$$

Here, the real-valued frequency response $H_r(\omega) \in R$ is given by

$$H_r(\omega) = 2\sum_{k=0}^{\frac{M-3}{2}} h(k)\sin\omega\left(\frac{M-1}{2}-k\right)$$

$$H_r(0) = 2\sum_{k=0}^{\frac{M}{2}-1} h(k)\sin 0\left(\frac{M-1}{2}-k\right) = 0$$

Then,

$$H(e^{j\omega}) = e^{-j\omega\frac{M-1}{2}+j\frac{\pi}{2}} H_r(\omega) = \begin{cases} |H_r(\omega)|e^{-j\omega\frac{M-1}{2}+j\frac{\pi}{2}} & for\, H_r(\omega) >= 0 \\ |H_r(\omega)|e^{-j\omega\frac{M-1}{2}+j\frac{3\pi}{2}} & for\, H_r(\omega) < 0 \end{cases}$$

$$|H(e^{j\omega})| = |H_r(\omega)|$$

$$\phi(\omega) = \begin{cases} -\omega\frac{M-1}{2} + \frac{\pi}{2} & for\, H_r(\omega) >= 0 \\ -\omega\frac{M-1}{2} + \frac{3\pi}{2} & for\, H_r(\omega) < 0 \end{cases}$$

3.2.5. Summary

M : *even /odd*	***Symmetry of impulse response***	$H_r(\omega) \in R$	$H_r(0)$
Even	Symmetrical	$H_r(\omega) = 2\sum_{k=0}^{\frac{M}{2}-1} h(k)\cos\omega\left(\frac{M-1}{2}-k\right)$	$H_r(0) = 2\sum_{k=0}^{\frac{M}{2}-1} h(k)$
Odd	Symmetrical	$H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{k=0}^{\frac{M-3}{2}} h(k)\cos\omega\left(\frac{M-1}{2}-k\right.$	$H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{k=0}^{\frac{M-3}{2}} h(k)$
Even	Antisymmetrical	$H_r(\omega) = 2\sum_{k=0}^{\frac{M}{2}-1} h(k)\sin\omega\left(\frac{M-1}{2}-k\right)$	$H_r(0) = 0$
Odd	Antisymmetrical	$H_r(\omega) = 2\sum_{k=0}^{\frac{M-3}{2}} h(k)\sin\omega\left(\frac{M-1}{2}-k\right)$	$H_r(0) = 0$

M *:even /odd*	***Symmetry of impulse response***	$\phi(\omega)$
Even	Symmetrical	$\phi(\omega) = \begin{cases} -\omega\frac{M-1}{2} & for\ H_r(\omega) >= 0 \\ -\omega\frac{M-1}{2} - \pi & for\ H_r(\omega) < 0 \end{cases}$
Odd	Symmetrical	$\phi(\omega) = \begin{cases} -\omega\frac{M-1}{2} & for\ H_r(\omega) >= 0 \\ -\omega\frac{M-1}{2} - \pi & for\ H_r(\omega) < 0 \end{cases}$
Even	Antisymmetrical	$\phi(\omega) = \begin{cases} -\omega\frac{M-1}{2} + \frac{\pi}{2} & for\ H_r(\omega) >= 0 \\ -\omega\frac{M-1}{2} + \frac{3\pi}{2} & for\ H_r(\omega) < 0 \end{cases}$
Odd	Antisymmetrical	$\phi(\omega) = \begin{cases} -\omega\frac{M-1}{2} + \frac{\pi}{2} & for\ H_r(\omega) >= 0 \\ -\omega\frac{M-1}{2} + \frac{3\pi}{2} & for\ H_r(\omega) < 0 \end{cases}$